Problem 6.40 and 6.41 Kleppner and Kolenkow
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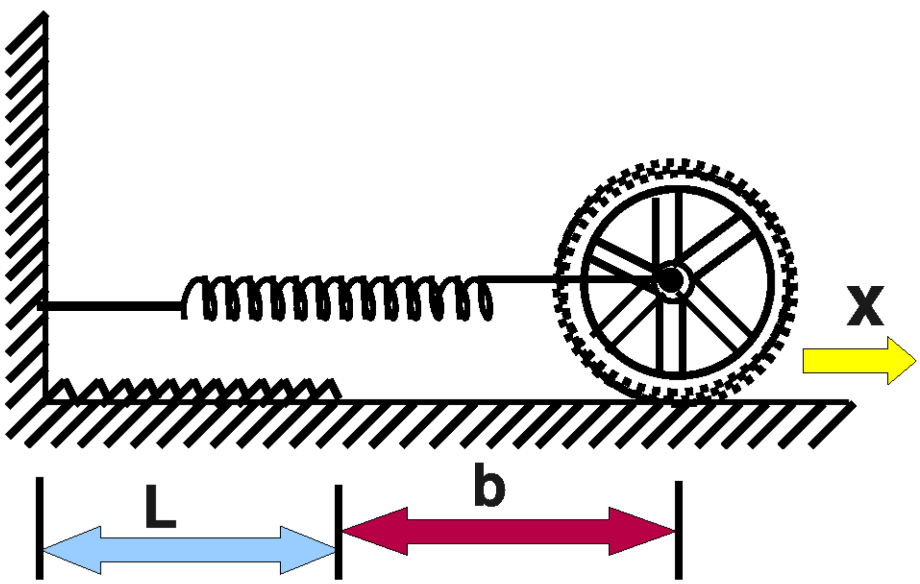
6.40 A wheel with fine teeth is attached to the end of a spring with constant k and unstretched length l. For $x > l$, the wheel slips freely on the surface, but for $x < l$ the teeth mesh with the teeth on the ground so that it cannot slip. Assume that all the mass of the wheel is in its rim.

a. The wheel is pulled to $x = l + b$ and released. How close will it come to the wall on its first trip?

b. How far out will it go as it leaves the wall?

c. What happens when the wheel next hits the gear track?



Sol. 6.40 **Top-down approach:** In order to know how close will it come to the wall, you need to know, what is the energy it starts with from the equilibrium point at the begining of gear track. A few things to note:

1. There is no dissipation to the right of gear track as there is no friction. If it approaches gear track with velocity v_0,

$$\frac{1}{2}kb^2 = \frac{1}{2}mv_0^2 \Rightarrow v_0 = b\sqrt{\frac{m}{k}} \tag{1}$$

2. Once on the gear-track it rolls without slipping and hence there is no dissipation.
3. When it hits the gear-track there is an impulsive force due to small protruding structure of gear-track(the surface prior to it was smooth) and hence there is a change in momentum.

$$\Delta p = m(v' - v_0) = f\Delta t \tag{2}$$

Here f is the force of friction and v' is the velocity with which it starts the journey on gear-track. To obtain v', note that right when it reaches the gear-track the torque τ about the point of contact is zero, as friction and gravity are both passing through the point of contact and spring force is zero at the equilibrium position. Thus total angular momentum about point of contact is conserved. Now according to eq. 6.13 (K&K),

$$L_z = I_0\omega + (R \times MV)_z \tag{3}$$

That is, L about an axis through some origin is equal to L about an axis passing through c.m. + L of the c.m. about the origin (here the point of contact). Thus L about point of contact immedeatly before and after it reaches gear:

$$\begin{aligned} L_i &= L_f \\ mv_0R &= I\omega + mv'R \qquad v' = R\omega \end{aligned}$$

Note that once it is on the gear track it has developed spin and hence $\omega = v'/R$ (rolling w/o slipping). From the above equation we get $v' = v_0/2$.

So it starts it's journey on gear track with energy $\frac{1}{2}mv'^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mv_0^2\right)$. There is a clear loss of $\frac{1}{4}mv_0^2$ which is dissipated when it encounters the gear. From the work-energy theorem,

$$\Delta K = \frac{1}{4}mv_0^2 = \int f dx = W \tag{4}$$

Alternatively,

$$\begin{aligned} \Delta p = m(v' - v_0) = m\frac{v_0}{2} &= \int F dt \\ m\frac{v_0}{2} \cdot v' = -m\frac{v_0^2}{4} &= \int F dt v' = \int P dt = W \end{aligned}$$

sol. a. Suppose it reaches distance a from the equilibrium position towards the wall.

$$\frac{1}{2}ka^2 = \frac{1}{2}mv'^2 + \frac{1}{2}I\omega^2 \Rightarrow a = \frac{b}{\sqrt{2}} \tag{5}$$

sol. b. To know how far away from the wall it goes on the return journey, we note that

- it reaches the equilibrium point with same energy it started from the equilibrium point as there is no dissipation on the gear track
- going past the gear track it will continue its spin motion as there is no torque that can change its spin angular momentum. Note that spin angular momentum is always about axis of rotation and hence spring force offers no torque and since there is no friction spin is conserved. Its linear velocity however will continuously decrease and hence it will slip (as $v < R\omega$) and its velocity will come to zero at the furthest point say c from the equilibrium point.

Thus from energy conservation:

$$\frac{1}{2}mv'^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}kc^2 \rightarrow c = \frac{b}{2} \tag{6}$$

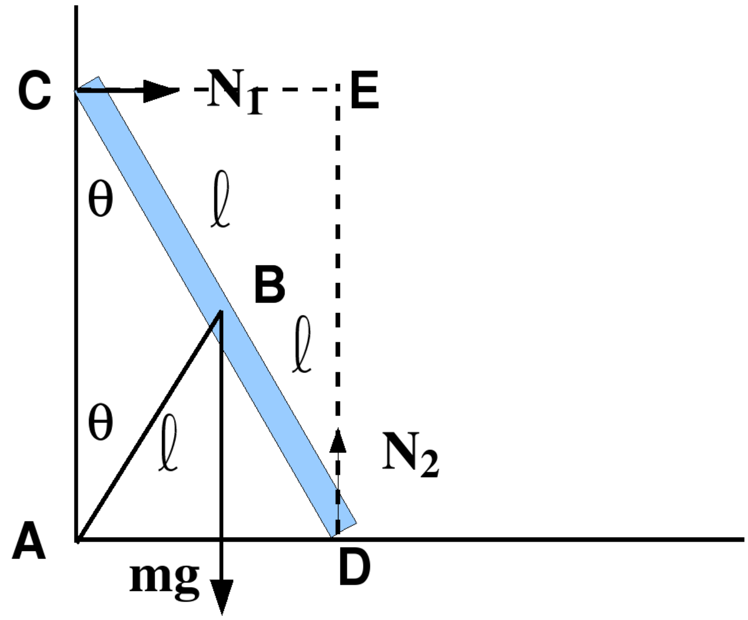
sol. c. Now when it begins its journey back toward equilibrium it is translating leftwards but spinning clockwise. The translational velocity of the c.m. increases and hence it gains orbital angular momentum in the counter-clockwise direction which becomes maximum and equals the spin angular momentum when it reaches the equilibrium poistion. That is

$$\begin{aligned} L &= L_{\text{orbital}} + L_{\text{spin}} \\ &= mR^2\omega - mR^2\omega \\ &= 0 \end{aligned}$$

Since there is no net torque about point of contact at equilibrium position, L is conserved and hence it is zero immediately after point of contact. Since wheel needs non-zero L in order to roll over gear-track it gets stuck at the gear track. Ofcourse all the remaining energy is dissipated.

If one instead considers torque about the axis of rotation passing through wheel's c.m. one finds that the torque due to force of enounter with gear about the c.m. produces spin angular momentum that is equal and opposite to the existing spin angular momentum and hence wheel stops spinning. Its linear motion is truncated and translational kinetic enegy $\frac{1}{4}mv_0^2$ is dissipated by the enounter with the gear track. Note that this was also the amount dissipated in the first encounter.

6.41 This problem utilizes most of the important laws introduced so far.
A plank of length 2ℓ leans against the wall. It starts to slip downwards without friction. Show that the top of the plank loses contact with the wall when it is two-thirds of its initial height.



Sol. 41 Shortest method: The most important thing to realize here is the trajectory of the center of mass of the plank before it leaves the contact with the wall. The center of mass follows a circle of radius ℓ. This follows if you look at the median to the hypotenuous which will be half the length of hypotenuous and hence l. Alternatively the coordinates of the center of mass are $x = l\sin\theta$ and $y = l\cos\theta$ and hence $x^2 + y^2 = l^2$.

The plank loses contact with the wall when the horizontal normal reaction N_1(and hence a_x) becomes negative. When a_x becomes negative v_x starts to decrease and hence we must look for the angle θ at which v_x is maximum; i.e., find $\frac{dv_x}{d\theta}$. v_x can be obtained from energy conservation.

Loss in P.E. = Gain in k.E.

$$mgl(1 - \cos\theta) = \frac{1}{2}ml\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 \tag{7}$$

A few imporant points to note here:

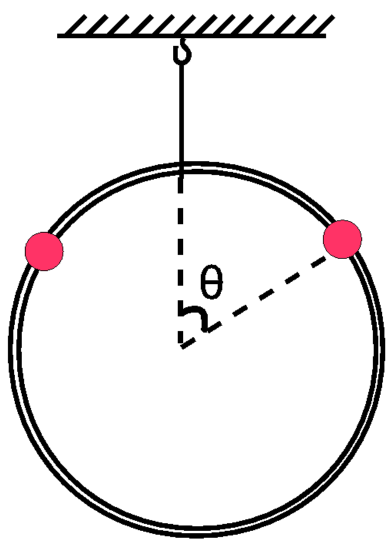
- The first term on the RHS is translational KE of of CM where $v_{cm} = l\dot{\theta}$.
- The second the rotation KE of plank. Here $I = m\frac{2\ell^2}{12} = \frac{1}{3}m\ell^2$. But we shall instead write $I = \beta m\ell^2$. The advantage is, we can easily the trace the shape dependence of the final answer in the terms of β and if β does not figure in the final answer then it means that the result is independent of the shape of the object (as it will actually be the case. In fact it is far more interesting than it appears.)
- Same $\dot{\theta}$ appears in both the terms. This is because the angle by which the plank rotates about the center of mass is equal to the angle by which the CM has itself moved on the ciruclar path.

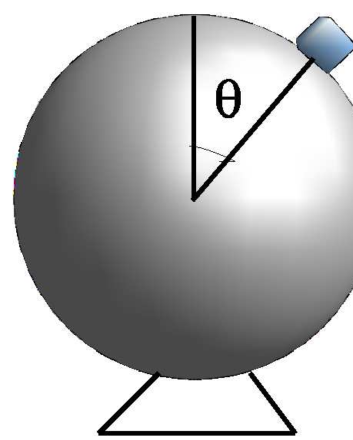
$$
\begin{aligned}
mgl(1-\cos\theta) &= \frac{1}{2}ml\dot{\theta}^2 + \frac{1}{2} + \beta m\ell^2\dot{\theta}^2 \\
v &= \sqrt{\frac{2g\ell}{1+\beta}}\sqrt{(1-\cos\theta)} \\
v_x &= \sqrt{\frac{2g\ell}{1+\beta}}\sqrt{(1-\cos\theta)}\cos\theta
\end{aligned}
$$

Find $\frac{dv_x}{d\theta}$ and equate it to zero to obtain:

$$
\cos\theta = \frac{2}{3} \tag{8}
$$

- This is indeed independent of β and hence an object of any other shape would loose contact with the wall at the same angle.
- The answer $\cos\theta = \frac{2}{3}$ should ring bells. Did you see the same answer in the contexts?





∗ **An interesting corelation:** In **Problem 4.6** You found that the cube loses the contact with the sphere at $\cos\theta = \frac{2}{3}$. In **Problem 4.7** you found that the normal reaction changes sign at $\cos\theta = \frac{2}{3}$. And now in the falling plank we again find that it loses contact with the wall at the same angle.
The differences in the three systems are all apparent. Once you strip them off the superficial differences it is the same system in different garbs. At the core, it is mass m moving on a circular trajectory constrained by the difference between centripetal force (a component of mg) and normal reaction. In the problem of falling plank the normal reaction forces N_1 and N_2 conspire to give a normal reaction $N = N_1 + N_2$ which can be considered to act on the

CM in a radially outward direction, in a manner very similar to the radially outwards normal reaction for the problem of cube sliding on a sphere and bead slide on a ring. The complications in the ring problem follow from the fact the bead threads through the ring and hence capable of pressing radially inwards as well as outwards and hence N has a scope of becoming negative. Nothing of the sorts happen in the case of cube sliding on a sphere and plank sliding on the wall and hence they loose contact. Note that after loosing the contact the trajectory of the CM of plank is no longer circular but a straight vertical line (see solved example 3.5 in K&K).

The Dynamical Method: Note that coordinates of CM are $(x, y) = (\ell \sin\theta, \ell \cos\theta)$. The plank loses contact when $N_1 = ma_x = m\ddot{x} = 0$. Let us find N_1 and N_2.

$$\begin{aligned} N_1 &= ma_x = m\ddot{x} = m\ell \cos\theta\ddot{\theta} - m\ell \sin\theta\dot{\theta}^2 \\ N_2 - mg &= ma_y = m\ddot{y} = mg - m\ell \sin\theta\ddot{\theta} - m\ell \cos\theta\dot{\theta}^2 \end{aligned}$$

So you need to know $\ddot{\theta}$ and $\dot{\theta}$. Former is obtained by torque analysis and latter is obtained by energy considerations.

In the torque analysis you may consider torque about any point keeping following things in mind.

- If the reference point is at rest (say points A or E), you may find torque due to all forces that are actually seen in the lab frame (gravity and normal reactions).
- If your reference point is accelarating then you may not or may need to take pseudo-forces in to account depending on whether the accelarating reference point is CM or not.

Torque about A: Since point A is at rest we do not need to consider pseudo forces. The torque equation is then:

$$\frac{dL_A}{dt} = \tau_A = 2N_2\ell \sin\theta - mg\ell \sin\theta - 2N_1\ell \cos\theta \tag{9}$$

L_A is the angular momentum about A and torque is positive when it is pointing out of the page. Point A being not the CM, the angular momentum about A is the sum of L of plank about CM and L of CM about about A.

$$L_A = -m\ell^2\dot{\theta} + \frac{m\ell^2\dot{\theta}}{3} = -\frac{2m\ell^2\dot{\theta}}{3} \tag{10}$$

A subtle point: Note that you could have obtained MI about point A using parallel axis theorem ($I_A = I_{cm} + m\ell^2 = \frac{4}{3}m\ell^2$) which is different from what we got. This would be true only if the the rotation of the CM about point of reference and the rotation about *the CM are of equal magnitude and sign. In our problem this is true only of points C and D but not points A and E and hence for point A we cannot use this theorem. As you can check we got a relative negative sign for the two contributions as well as difference in magnitudes.*

From the τ_A and N_1 and N_2, we can isolate $\ddot{\theta}$:

$$\ddot{\theta} = \frac{3g}{4l} \sin\theta \tag{11}$$

From the enegy conservation $mg\ell = T + V$ we can get $\dot{\theta}$:

$$\dot{\theta}^2 = \frac{3g}{2l}(1 - \cos\theta) \tag{12}$$

Substituting for $\dot{\theta}$ and $\ddot{\theta}$ in $N_1 = 0$ we get

$$\cos\theta = \frac{2}{3} \tag{13}$$

Torque about point B: Point B is accelarating, but since it is center of mass, we do not need to use pseudo-force. $L_B = I_{cm}\dot{\theta}$ and $I_{cm} = \frac{m(2\ell)^2}{12} = \frac{m\ell^2}{3}$

$$\begin{aligned} \frac{dL_B}{dt} = \frac{m\ell^2\ddot{\theta}}{3} &= \tau_B = N_2\ell\sin\theta - N_1\ell\cos\theta \\ &= mg\ell\sin\theta - m\ell^2\ddot{\theta} \end{aligned}$$

This again leads to same equation of motion.

This solution along with all the slides have been uploaded on intrabits as well as on personal webpage: http://discovery.bits-pilani.ac.in/discipline/physics/rishikesh/physics1.html